

Efficiency in Quantum Key Distribution protocols with entangled gaussian states

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Abstract. Quantum Key Distribution (QKD) refers to specific quantum strategies which permit the secure distribution of a key between two parties that wish to communicate secretly. Quantum Cryptography has proven unconditionally secure in ideal scenarios and has been successfully implemented using quantum states with finite (discrete) as well as infinite (continuous) degrees of freedom. In here, we analyze the efficiency of QKD protocols that use as a resource entangled gaussian states and gaussian operations only. In this framework, it has already been shown that QKD is possible [1] but the issue of efficiency has not been considered. We propose a figure of merit to relate the efficiency of the protocol to the entanglement and purity of the states.

1. Introduction

Quantum cryptography relies on the possibility of establishing a secret random key between two distant parties traditionally denoted by Alice and Bob. If the key is securely distributed, the algorithms used to encode and decode any message can be made public without compromising security. The key consists typically in a random sequence of bits which both, Alice and Bob, share as a string of classically correlated data. The superiority of quantum cryptography comes from the fact that the laws of quantum mechanics permit to the legitimate users (Alice and Bob) to infer if an eavesdropper has monitored the distribution of the key and has gained information about it. If this is the case, Alice and Bob will both agree in withdrawing the key and will start the distribution of a new one. In contrast, classical key distribution, no matter how difficult the distribution from

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a technological point of view is, can always be intercepted by an eavesdropper without Alice and Bob realizing it. In quantum cryptography, there exist several protocols that Alice and Bob can use in order to establish a secret key. Some of them, like Ekert91, use as a resource shared entanglement between the two parties [2], while in others like BB84 [3] the key is established by sending non entangled quantum states between the parties and communicating classically. If Alice and Bob share a collection of distillable entangled states, they can always obtain from them a smaller number of maximally entangled states from which they can establish a secure key [4]. The number of singlets (maximally entangled states) that can be extracted from a quantum state using only Local Operations and Classical Communication (LOCC) is referred to as the Entanglement of Distillation E_D . For establishing a key, another important concept is the number of secret bits K_D , that can be extracted from a quantum state using LOCC. Since a secret bit can always be extracted from maximally entangled state, $E_D \leq K_D$. There exist also quantum states which are entangled but cannot be distilled, *i.e.*, have $E_D = 0$. They are usually referred to as bound entangled states since its entanglement is bound to the state. Nevertheless, for some of those states it has been shown that $K_D \neq 0$ and thus they can be used to establish a secret key [5].

A particular case of states that cannot be “distilled” by “normal” procedures are continuous variables gaussian states, *e.g.*, thermal, coherent and squeezed states of light. By “normal” procedures we mean operations that preserve the gaussian character of the state (gaussian operations). They correspond *e.g.* to beam splitters, squeezers, mirrors, etc. Thus, in the gaussian scenario all entangled gaussian states posses bound entanglement. Quantum cryptography with gaussian states using gaussian operations has been experimentally implemented using “prepare and measure schemes” with either squeezed or coherent states [6, 7]. Those schemes do not demand entanglement between the parties.

Recently, Navascués *et al.* [1] have shown that it is also possible with only gaussian operations to extract a secret key *à la* Ekert91 from entangled gaussian states, in spite the fact that these states are not distillable. In other words, it has been shown that in the gaussian scenario all entangled gaussian states fulfill $GK_D > 0$ (where the letter G stands for gaussian) while $GE_D = 0$. Alice and Bob can extract a list of classically correlated bits from a set a of 1×1 entangled modes as follows: i) they agree on a value $x_0 > 0$, ii) Alice(Bob) measures the quadrature of each of her(his) modes $\hat{X}_A(\hat{X}_B)$, iii) they accept only outputs such that $|x_A| = |x_B| = x_0$, iv) they associate *e.g.*, the classical value 0(1) to $x_i = +x_0(-x_0)$, $i = A, B$ and thus establish a list of classically correlated bits. From there, they can apply Advantage Distillation [8] to establish the secret key. This protocol is secure against individual eavesdropper attacks. Since the protocol is based on output coincidences of the measurements of the quadratures which, by definition, are operators with a continuous spectrum, the protocol has zero efficiency [1].

Here we study the consequences of relaxing the above condition to a more realistic scenario. We assume that Alice and Bob can extract a list of sufficiently correlated classical bits obtained by accepting measurement outputs that do not coincide but are bound within a range. We ask ourselves which is the possibility that Alice and Bob can still distribute the key in a secure way under individual and coherent attacks. We obtain that there exists always a finite interval for which the protocol can be implemented successfully. The length of this interval depends on the entanglement and on the purity of the shared states, and increases with increasing entanglement.

The paper is organised as follows. In Section 2. we present the formalism needed to tackle this problem. In Section 3. we first review the previous protocol [1] and present our new results. Finally, we present our conclusions in Section 4.

2. Formalism

Systems of continuous variables are often expressed in terms of modes, where each mode has two associated canonical degrees of freedom (“position” and “momentum”) which fulfill the canonical commutation relations (CCR). The CCR for a quantum system with n modes can be compactly expressed *via* the symplectic matrix. Denoting the canonical coordinates by $\hat{\mathbf{R}}^T = (\hat{q}_1, \hat{p}_1, \dots, \hat{q}_n, \hat{p}_n) = (\hat{R}_1, \dots, \hat{R}_{2n})$ the CCR simply read $[\hat{R}_i, \hat{R}_j] = i(J_n)_{ij}$, where $i, j = 1, \dots, 2n$ and

$$\mathbf{J}_n = \bigoplus_{i=1}^n \mathbf{J}, \quad \mathbf{J} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (1)$$

The matrix \mathbf{J} defines the symplectic scalar product and describes the geometry of the phase space. There is a bijective map between a quantum state described by a density matrix $\hat{\rho}$ (in an infinite Hilbert space) and its corresponding characteristic function χ_ρ , which is given by the Fourier-Weyl transform:

$$\chi_\rho(\zeta) \equiv \text{tr}\{\hat{\rho}\hat{W}(\zeta)\}, \quad (2)$$

$$\hat{\rho} \equiv \frac{1}{(2\pi)^n} \int d^{2n}\zeta \chi_\rho(\zeta) \hat{W}(-\zeta), \quad (3)$$

where $\zeta \in \mathbb{R}^{2n}$ and $\hat{W}_\zeta = e^{i\zeta^T \mathbf{J}_n \hat{\mathbf{R}}}$ are the so-called Weyl operators. Gaussian states are characterized by a gaussian χ_ρ function,

$$\chi_\rho(\zeta) = e^{i\zeta^T \cdot \mathbf{J}_n \cdot \mathbf{d} - \frac{1}{4} \zeta^T \mathbf{J}_n^T \cdot \boldsymbol{\gamma} \cdot \mathbf{J}_n \zeta}, \quad (4)$$

where \mathbf{d} is a $2n$ real vector, called displacement vector (DV), and $\boldsymbol{\gamma}$ is a $2n \times 2n$ symmetric real matrix, denoted as covariance matrix (CM). A convenient representation of gaussian quantum states is given in terms of the Wigner quasi-distribution

function \mathcal{W}_ρ [9], which is related to the characteristic function by the symplectic Fourier transform which preserves the gaussian character,

$$\mathcal{W}_\rho(\zeta) = \frac{1}{(2\pi)^{2n}} \int d^{2n}\boldsymbol{\eta} \chi_\rho(\boldsymbol{\eta}) e^{-i\boldsymbol{\eta}^T \cdot \mathbf{J}_n \cdot \zeta}, \quad (5)$$

$$\chi_\rho(\boldsymbol{\eta}) = \int d^{2n}\zeta \mathcal{W}_\rho(\zeta) e^{i\boldsymbol{\eta}^T \cdot \mathbf{J}_n \cdot \zeta}, \quad (6)$$

where $\boldsymbol{\eta} \in \mathbb{R}^{2n}$. Thus, a gaussian quantum state can equivalently be defined as a quantum state whose Wigner function is gaussian,

$$\mathcal{W}_\rho(\zeta) = \frac{1}{\pi^n \sqrt{\det \boldsymbol{\gamma}}} e^{-(\zeta - \mathbf{d})^T \cdot \frac{1}{\boldsymbol{\gamma}} \cdot (\zeta - \mathbf{d})}. \quad (7)$$

\mathbf{d} and $\boldsymbol{\gamma}$ are defined as:

$$d_i = \text{tr}(\hat{\rho} \hat{R}_i), \quad (8)$$

$$\gamma_{ij} = \text{tr}(\hat{\rho} \{\hat{R}_i - d_i \hat{\mathbb{I}}, \hat{R}_j - d_j \hat{\mathbb{I}}\}), \quad (9)$$

and are computed *via* the first and second moments of the characteristic function,

$$d'_i = -i \frac{\partial}{\partial \zeta_i} \chi_\rho(\zeta) \Big|_{\zeta=0} = \text{tr}(\hat{\rho} \hat{R}'_i), \quad (10)$$

$$\frac{\gamma'_{ij}}{2} + d'_i d'_j = (-i)^2 \frac{\partial^2}{\partial \zeta_i \partial \zeta_j} \chi_\rho(\zeta) \Big|_{\zeta=0} = \frac{1}{2} \text{tr}(\hat{\rho} \{\hat{R}'_i, \hat{R}'_j\}), \quad (11)$$

where $\hat{R}'_i = J_{ij} \hat{R}_j$, $d'_i = J_{ij} d_j$ and $\gamma'_{ij} = J_{ik}^T \gamma_{kl} J_{lj}$.

In analogy with classical probability theory, the displacement vector \mathbf{d} plays the role of the mean value $\mu_i = \text{E}[x_i]$, and the covariance matrix elements $\boldsymbol{\gamma}$ plays the role of the covariances $C_{ij} = \text{Cov}(x_i, x_j) = \text{E}[(x_i - \mu_i)(x_j - \mu_j)]$ of a classical probability distribution. So only relative displacement vectors have physical meaning and only the non block-diagonal terms of the covariance matrix tell us about the quantum correlations present in the state.

Since the density matrix is a semidefinite positive operator, $\hat{\rho} \geq 0$, the corresponding covariance matrix must fulfill: $\boldsymbol{\gamma} + i\mathbf{J}_n \geq 0$. One can also define the fidelity between continuous gaussian states in terms of Wigner functions. We use here the Bures-Uhlmann fidelity between two arbitrary states $\hat{\rho}_1$ and $\hat{\rho}_2$ defined as $\mathcal{F}(\hat{\rho}_1, \hat{\rho}_2) = \left[\text{tr} \sqrt{\sqrt{\hat{\rho}_1} \hat{\rho}_2 \sqrt{\hat{\rho}_1}} \right]^2$ [10] which coincides with the so called Hilbert-Schmidt fidelity

$$\mathcal{F}(\hat{\rho}_1, \hat{\rho}_2) = \text{tr}(\hat{\rho}_1 \hat{\rho}_2), \quad (12)$$

whenever at least one of the states is pure. At the level of CM, using the Quantum Parseval relation [11], the Hilbert-Schmidt fidelity between two gaussian states can

be written as:

$$\begin{aligned}\mathcal{F}(\hat{\rho}_1, \hat{\rho}_2) &= (2\pi)^n \int d^{2n}\zeta \mathcal{W}_1(\zeta) \mathcal{W}_2(\zeta) = \\ &= \frac{1}{\sqrt{\det\left(\frac{\gamma_1 + \gamma_2}{2}\right)}} e^{-(\mathbf{d}_2 - \mathbf{d}_1)^T \cdot \left(\frac{1}{\gamma_1 + \gamma_2}\right) \cdot (\mathbf{d}_2 - \mathbf{d}_1)}\end{aligned}\quad (13)$$

where $\gamma_{1(2)}$ and $\mathbf{d}_{1(2)}$ belong to $\hat{\rho}_{1(2)}$. Clearly, only relative DVs are of physical significance. Regarding the purity of the state, *i.e.* $\mathcal{P}(\gamma) = \text{tr}(\hat{\rho}^2) = \det(\gamma)^{-1/2} \leq 1$. It is important to notice that gaussian states always admit a purification. Thus, any mixed gaussian state of n modes can be expressed as the reduction of a pure gaussian state of $2n$ modes of the form:

$$\gamma_{2n} = \begin{pmatrix} \gamma_n & \mathbf{C}_n \\ \mathbf{C}_n^T & \boldsymbol{\theta}_n \gamma_n \boldsymbol{\theta}_n^T \end{pmatrix}, \quad \mathbf{C}_n = \mathbf{J}_n \sqrt{-(\mathbf{J}_n \gamma_n)^2 - \mathbb{I}} \boldsymbol{\theta}_n, \quad \boldsymbol{\theta}_n = \bigoplus_{i=1}^n \boldsymbol{\theta},$$

such that the mixed state can be obtained after tracing out n modes from γ_{2n} . Here $\boldsymbol{\theta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which is the momentum reflection in phase-space, is the associated symplectic operation.

For what follows it is also important to study entanglement properties in the formalism of covariance matrices. A necessary and sufficient condition for separability for 1×1 and $1 \times N$ modes of bipartite gaussian states is provided by the PPT criterion [12] (for the rest of the states this criterion is only necessary but not sufficient). The PPT criterion tells us that a state is entangled if and only if the state $\hat{\rho}$ has non positive partial transposition (NPPT): $\hat{\rho}^{T_A} < 0$. In terms of CMs this criterion reads $\boldsymbol{\theta}_A \gamma \boldsymbol{\theta}_A^T + i\mathbf{J} < 0$. The second property, particularly relevant in what follows, is that any NPPT gaussian state can be mapped by Gaussian Local Operations and Classical Communication (GLOCC) to an NPPT symmetric state of 1×1 modes.

To calculate the entanglement of our states, we will use the logarithmic negativity as entanglement measure; $\text{LN}(\hat{\rho}) = \log_2 \|\hat{\rho}^{T_A}\|_1$, where $\|\hat{A}\|_1 = \text{tr} \sqrt{\hat{A}^\dagger \hat{A}}$ can be easily computed through the sum of the singular values of \hat{A} . One can extend this measure to gaussian states of n modes through [13]:

$$\text{LN}(\gamma_n) = - \sum_{i=1}^n \log_2 \min(\tilde{\mu}_i, 1) \quad (14)$$

where $\{\pm \tilde{\mu}_i\} = \text{spec}(-i\mathbf{J}_n \gamma_n^{T_A})$ *i.e.*, the symplectic spectrum of the partial transposed CM.

With this formalism at hand we now move to the presentation of our calculations and results.

3. Results

First, we summarize the main steps of the protocol used in [1]. Without losing generality, and by virtue of the properties of gaussian states, one should only consider the case in which Alice and Bob share many copies of a quantum system of 1×1 symmetric NPPT gaussian state $\hat{\rho}_{AB}$. To extract a list of classically correlated bits to establish a secret key, each party measures the quadratures of her/his mode $\hat{X}_{A,B}$ and accepts only those outputs $x_{A,B}$ for which both parties have a consistent result $|x_A| = |x_B| = x_0$. With probability $p(i, j)$, each party associates the classical bit $i = 0(1)$ to her/his outcome $+x_0(-x_0)$. The probability that their symbols do not coincide is given by $\epsilon_{AB} = (\sum_{i \neq j} p(i, j)) / (\sum_{i, j} p(i, j))$. Having fixed a string of N classical correlated values, they can apply classical advantage distillation [8]. To this aim, Alice generates a random bit b and encodes her string of N classical bits into a vector \vec{b} of length N such that $b_{Ai} + b_i = b \bmod 2$. Bob checks that for his symbols all results $b_{Bi} + b_i = b' \bmod 2$ are consistent, and in this case accepts the bit b . The new error probability is given by

$$\epsilon_{AB,N} = \frac{(\epsilon_{AB})^N}{(1 - \epsilon_{AB})^N + (\epsilon_{AB})^N} \leq \left(\frac{\epsilon_{AB}}{1 - \epsilon_{AB}} \right)^N, \quad (15)$$

which tends to zero for sufficiently large N . The most general scenario for eavesdropping is to assume that Eve has access to the states before their distribution. Hence, the states that Alice and Bob share correspond to the reduction of a pure 4-mode state. Now security with respect to individual attacks from the eavesdropper Eve, can be established if

$$\left(\frac{\epsilon_{AB}}{1 - \epsilon_{AB}} \right)^N < |\langle e_{++} | e_{--} \rangle|^N, \quad (16)$$

where $|e_{\pm\pm}\rangle$ denotes the state of Eve once Alice and Bob have projected their states onto $|\pm x_0\rangle$. Notice that Eve can gain information if the overlap between her states after Alice and Bob have measured coincident results is sufficiently small. The above inequalities come from the fact that in the case of individual attacks the error on Eve's estimation of the final bit b is bound from below by a term proportional to $|\langle e_{++} | e_{--} \rangle|^N$ [1]. Therefore, Alice and Bob can establish a key if

$$\frac{\epsilon_{AB}}{1 - \epsilon_{AB}} < |\langle e_{++} | e_{--} \rangle|. \quad (17)$$

In [1] it was shown that any 1×1 NPPT state fulfills the above inequality and thus any NPPT gaussian state can be used to establish a secure key in front of individual eavesdropper attacks.

Let us now present our results. Notice that since security relies on the fact that Alice and Bob have better correlations than the information the eavesdropper can

learn about their state, perfect correlation is not a requirement to establish a secure key. We denote Alice's outputs by x_{0A} and we calculate which are the outputs Bob can accept so that the correlation established between Alice and Bob outputs can be used to extract a secret bit.

We use the standard form of a bipartite 1×1 mode gaussian state,

$$\gamma_{AB} = \begin{pmatrix} \lambda_A & 0 & c_x & 0 \\ 0 & \lambda_A & 0 & -c_p \\ c_x & 0 & \lambda_B & 0 \\ 0 & -c_p & 0 & \lambda_B \end{pmatrix} \quad (18)$$

with $\lambda_{A,B} \geq 0$, and $c_x \geq c_p \geq 0$ (we can shift the DV to 0). The gaussian state is called symmetric if $\lambda_A = \lambda_B = \lambda$ and fully symmetric if also $c_x = c_p$. We shall deal with mixed symmetric states. The positivity condition reads $(\lambda - c_x)(\lambda + c_p) \geq 1$, while the entanglement NPPT condition is given by $(\lambda - c_x)(\lambda - c_p) < 1$. As in [1], we impose that the global state including Eve is pure (she has access to all degrees of freedom outside Alice and Bob) while the mixed symmetric state, shared by Alice and Bob is just its reduction,

$$\gamma_{ABE} = \begin{pmatrix} \gamma_{AB} & C \\ C^T & \theta \gamma_{AB} \theta^T \end{pmatrix}, \quad (19)$$

$$C = J_{AB} \sqrt{-(J_{AB} \gamma_{AB})^2 - \mathbb{I}_2} \theta_{AB} = \begin{pmatrix} 0 & -X & 0 & -Y \\ -X & 0 & -Y & 0 \\ 0 & -Y & 0 & -X \\ -Y & 0 & -X & 0 \end{pmatrix}, \quad (20)$$

$$\theta_{AB} = \theta_A \oplus \theta_B, \quad J_{AB} = J_A \oplus J_B, \quad (21)$$

where

$$X = \frac{\sqrt{a+b} + \sqrt{a-b}}{2},$$

$$Y = \frac{\sqrt{a+b} - \sqrt{a-b}}{2},$$

and $a = \lambda^2 - c_x c_p - 1$, $b = \lambda(c_x - c_p)$.

Performing a measurement with uncertainty σ , the probability that Alice finds $\pm|x_{0A}|$ while Bob finds $\pm|x_{0B}|$, is given by the overlap between the state of Alice and Bob, $\hat{\rho}_{AB}$, and a pure product state $\hat{\rho}_{A,i} \otimes \hat{\rho}_{B,j}$ (with $i, j = 0, 1$) of gaussians centered at $\pm|x_{0A}|(\pm|x_{0B}|)$ respectively with σ width (notice $\hat{\rho}_{A,0} \equiv | +|x_{0A}| \rangle \langle +|x_{0A}| |$). We use here the Hilbert-Schmidt fidelity for calculation, which

gives:

$$\begin{aligned}
p(0,0) &= p(1,1) = \text{tr}[\hat{\rho}_{AB}(\hat{\rho}_{A,0} \otimes \hat{\rho}_{B,0})] = \\
&= (2\pi)^4 \int d^4 \zeta_{AB} \mathcal{W}_{\rho_{AB}}(\zeta_{AB}) \mathcal{W}_{\rho_{A,0} \otimes \rho_{B,0}}(\zeta_{AB}) = \\
&= K(\sigma) \exp \left(\frac{2|x_{0A}||x_{0B}|c_x - (\lambda + \sigma^2)(x_{0A}^2 + x_{0B}^2)}{(\lambda + \sigma^2)^2 - c_x^2} \right), \tag{22}
\end{aligned}$$

for the probability that their symbols do coincide and,

$$p(0,1) = p(1,0) = K(\sigma) \exp \left(\frac{-2|x_{0A}||x_{0B}|c_x - (\lambda + \sigma^2)(x_{0A}^2 + x_{0B}^2)}{(\lambda + \sigma^2)^2 - c_x^2} \right), \tag{23}$$

for the probability that they do not coincide, where

$$K(\sigma) = \frac{4\sigma^2}{\sqrt{(\lambda + \sigma^2)^2 - c_x^2} \sqrt{(\lambda\sigma^2 + 1)^2 - c_p^2\sigma^4}}. \tag{24}$$

Their error probability for $\sigma \rightarrow 0$ reads

$$\epsilon_{AB} = \lim_{\sigma \rightarrow 0} \frac{\sum_{i \neq j} p(i,j)}{\sum_{i,j} p(i,j)} = \frac{1}{1 + \exp \left(\frac{4c_x|x_{0A}||x_{0B}|}{\lambda^2 - c_x^2} \right)}. \tag{25}$$

Let us calculate the state of Eve $|e_{\pm\pm}\rangle$ after Alice has projected onto $|\pm|x_{0A}\rangle$ and Bob onto $|\pm|x_{0B}\rangle$:

$$\gamma_{++} = \gamma_{--} = \begin{pmatrix} \gamma_x & 0 \\ 0 & \gamma_x^{-1} \end{pmatrix}, \quad \gamma_x = \begin{pmatrix} \lambda & c_x \\ c_x & \lambda \end{pmatrix}, \tag{26}$$

$$d_{\pm\pm} = \mp \begin{pmatrix} 0 \\ 0 \\ A\delta x_0 - B\Delta x_0 \\ A\delta x_0 + B\Delta x_0 \end{pmatrix}, \tag{27}$$

where $A = \frac{\sqrt{a+b}}{\lambda+c_x}$, $B = \frac{\sqrt{a-b}}{\lambda-c_x}$, $\Delta x_0 = |x_{0B}| - |x_{0A}|$ and $\delta x_0 = |x_{0B}| + |x_{0A}|$. The overlap between the two states of Eve is given by:

$$\begin{aligned}
|\langle e_{++}|e_{--}\rangle|^2 &= \exp \left(\frac{-4}{\lambda^2 - c_x^2} \left[\left(\frac{x_{0A}^2 + x_{0B}^2}{2} \right) (\lambda^2 - c_x^2 - 1)\lambda + \right. \right. \\
&\quad \left. \left. + |x_{0A}||x_{0B}|(c_x - c_p(\lambda^2 - c_x^2)) \right] \right). \tag{28}
\end{aligned}$$

Substituting Eqs. (25) and (28) into (17) one can check, after some algebra, that the last inequality reduces to:

$$\left(\frac{x_{0A}^2 + x_{0B}^2}{2}\right)(\lambda^2 - c_x^2 - 1)\lambda + |x_{0A}||x_{0B}|(-c_x - c_p(\lambda^2 - c_x^2)) < 0. \quad (29)$$

Notice that condition (29) imposes both, restrictions on the parameters defining the state (λ, c_x, c_p) , and on the outcomes of the measurements (x_{0A}, x_{0B}) . The constraints on the state parameters are equivalent to demand that the state is NPPT and satisfies

$$(\lambda - c_x)(\lambda + c_x) \geq 1. \quad (30)$$

Nevertheless, as $c_x \geq c_p$, any positive state fulfills this condition. Hence for any NPPT symmetric state, there exists, for a given x_{0A} , a range of values of x_{0B} such that secret bits can be extracted (Eq. (17) is fulfilled). This range is given by

$$\Delta x_0 = |x_{0B}| - |x_{0A}| \in \mathfrak{D}_\alpha = \left[\frac{2}{-\sqrt{\alpha} - 1}, \frac{2}{\sqrt{\alpha} - 1} \right] |x_{0A}|, \quad (31)$$

where

$$\alpha = \left(\frac{c_x - \lambda}{c_x + \lambda} \right) \left[\frac{1 - (\lambda + c_x)(\lambda + c_p)}{1 - (\lambda - c_x)(\lambda - c_p)} \right]. \quad (32)$$

After Alice communicates $|x_{0A}|$ to Bob, he will accept only measurement outputs within the above interval. The interval Δx_0 is well defined if $\alpha \geq 1$, which equval to fulfill Eq. (30). Notice also that the interval is not symmetric around $|x_{0A}|$ because the probabilities calculated in Eqs. (22) and (23) do depend on this value in a non symmetric way. The length of the interval of valid measurements outputs for Bob is given by

$$D_\alpha = \frac{4\sqrt{\alpha}}{\alpha - 1} |x_{0A}|. \quad (33)$$

It can be observed that maximal $D_\alpha \rightarrow \infty$ ($\alpha = 1$) corresponds to the case when Alice and Bob share a pure state (Eve is disentangled from the system) and thus condition (17) is always fulfilled. On the other hand, any mixed NPPT symmetric state ($\alpha > 1$) admits a finite D_α . This ensures a *finite* efficiency on establishing a secure secret key in front of individual attacks.

If we assume that Eve performs more powerful attacks, namely finite coherent attacks, then security is only guaranteed if [1]:

$$\frac{\epsilon_{AB}}{1 - \epsilon_{AB}} < |\langle e_{++} | e_{--} \rangle|^2. \quad (34)$$

This condition is more restrictive than (17). With a similar calculation as before we obtain that now security is not guaranteed for all mixed entangled symmetric NPPT states, but only for those that also satisfy:

$$\lambda - (\lambda + c_x)(\lambda - c_x)(\lambda - c_p) > 0. \quad (35)$$

For such states, and given a measurement result x_{0A} of Alice, Bob will only accept outputs within the range:

$$\Delta x_0 = |x_{0B}| - |x_{0A}| \in \mathfrak{D}_\beta = \left[\frac{2}{-\sqrt{\beta}-1}, \frac{2}{\sqrt{\beta}-1} \right] |x_{0A}|, \quad (36)$$

where

$$\beta = \frac{2\lambda(\lambda^2 - c_x^2 - 1)}{\lambda - (\lambda + c_x)(\lambda - c_x)(\lambda - c_p)} \geq 1. \quad (37)$$

As before, $\beta \geq 1$ is fulfilled by (30) and (35).

Let us now focus on the efficiency issue. We define the efficiency $E(\gamma_{AB})$ of the protocol for a given state γ_{AB} , as the average probability of obtaining a classically correlated bit. Explicitly,

$$E(\gamma_{AB}) = \int_{\Delta x_0 \in \mathfrak{D}} dx_{0A} dx_{0B} (1 - \epsilon_{AB}) \text{tr}(\hat{\rho}_{AB}|x_{0A}, x_{0B}\rangle\langle x_{0A}, x_{0B}|). \quad (38)$$

The marginal distribution in phase-space is easily computed by integrating the corresponding Wigner function in momentum space [14]:

$$\begin{aligned} \text{tr}(\hat{\rho}_{AB}|x_{0A}, x_{0B}\rangle\langle x_{0A}, x_{0B}|) &= \int \int dp_A dp_B \mathcal{W}_{\rho_{AB}}(\zeta_{AB}) = \\ &= \frac{\exp\left(\frac{2c_x x_{0A} x_{0B} - \lambda(x_{0A}^2 + x_{0B}^2)}{\lambda^2 - c_x^2}\right)}{\pi \sqrt{\lambda^2 - c_x^2}}, \end{aligned} \quad (39)$$

but the final expression of (38) has to be calculated numerically. Note that if Alice and Bob share as a resource N identical states (NPPT state for individual attacks, and NPPT fulfilling (35) for finite coherent attacks), the number of classically correlated bits that can be extracted from them is $\sim N \times E(\gamma_{AB})$. The efficiency (38) increases with increasing D and decreasing ϵ_{AB} . In particular, for the protocol given in [1], $D = 0$, and therefore $E(\gamma_{AB}) = 0$ for any state.

We investigate now the dependence of $E(\gamma_{AB})$ with the entanglement of the NPPT mixed symmetric state used for the protocol as well as with the purity of the state. As a measure of the entanglement between Alice and Bob we compute the logarithmic negativity

$$\text{LN}(\gamma_{AB}) = \log_2 \left(\frac{1}{\sqrt{(\lambda - c_x)(\lambda - c_p)}} \right) > 0. \quad (40)$$

In Fig. 1, we display the efficiency of the protocol (assuming individual attacks) versus entanglement shared between Alice and Bob for different states γ_{AB} . There is not a one-to-one correspondence between $E(\gamma_{AB})$ and entanglement, since states

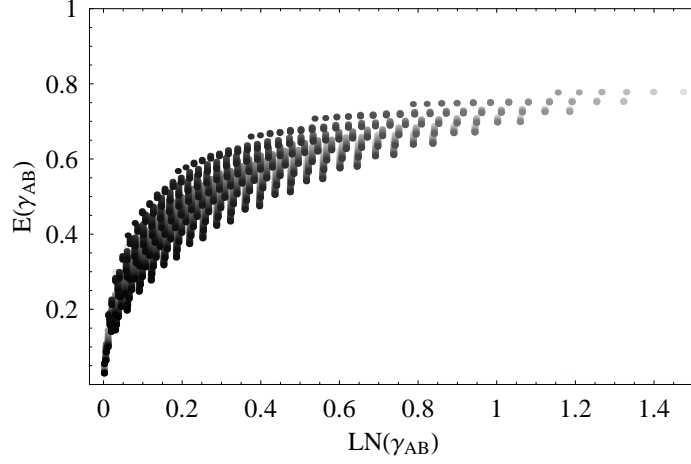


Fig. 1: Protocol efficiency (quantified by $E(\gamma_{AB})$) versus the entanglement measured by logarithmic negativity $\text{LN}(\gamma_{AB})$. The shading from black to white corresponds to purity from zero to one.

with the same entanglement can have different purity, which can lead to different efficiency. This is so because there are two favorable scenarios to fulfill Eq. (17). The first one is to demand large correlations so that the relative error ϵ_{AB} of Alice and Bob is small. The second scenario happens when Alice and Bob share a state with high purity, *i.e.*, Eve is very disentangled. In this case, independently of the error ϵ_{AB} , Eq. (17) can be fulfilled more easily.

Despite the fact that efficiency generally increases with increasing entanglement, this enhancement, as depicted in the figure, is a complex function of the parameters involved. Nevertheless, one can see that there exist an entanglement threshold (around $\text{LN}(\gamma_{AB}) \simeq 0.2$) below which the protocol efficiency diminishes drastically no matter how mixed are the states shared between Alice and Bob.

It is also illustrative to examine the dependence of α (which determines the interval length D_α) on the entanglement of the states shared by Alice and Bob. In Fig. 2 we plot the logarithmic negativity of a given state versus the parameter α . States with the same entanglement but different purity are associated to quite different values of α . Nevertheless states with high entanglement permit a large interval length (small α) and, thus, high efficiency.

In both, Fig. 1 and Fig. 2, we have observed that states with different entanglement give the same efficiency. However it is important to point out that to extract the key's bits, classical advantage distillation [8] stills needs to be performed. The efficiency of Maurer's protocol, strongly increases with decreasing ϵ_{AB} , and, therefore, the states with higher entanglement will provide a higher key rate.

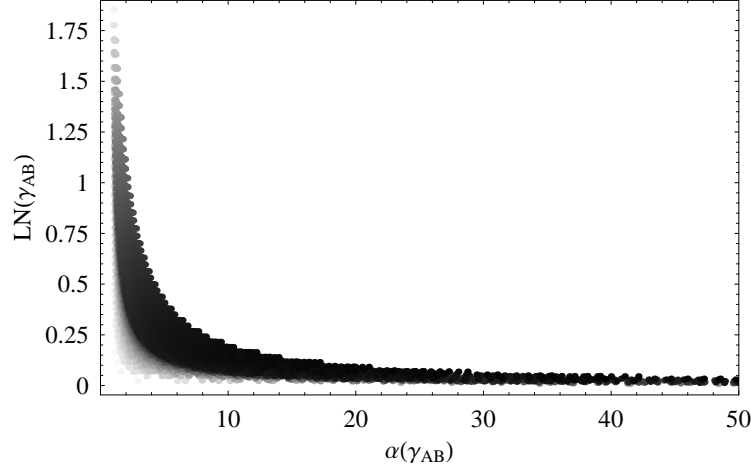


Fig. 2: Entanglement of the states shared between Alice and Bob measured in terms of the logarithmic negativity $\text{LN}(\gamma_{AB})$ as a function of the parameter $\alpha(\gamma_{AB})$ under individual attacks. The shading from black to white corresponds to purity from zero to one.

4. Conclusions

Efficiency is a key issue for any experimental implementation of quantum cryptography since available resources are not unlimited. Here, we have shown that the sharing of entangled gaussian variables and the use of only gaussian operations permits efficient quantum key distribution against individual and finite coherent attacks. All mixed NPPT symmetric states can be used to extract secret bits under individual attacks whereas under finite coherent attacks and additional condition has to be fulfilled. We have introduced a figure of merit (the efficiency E) to quantify the number of classical correlated bits that can be use to distill a key from a sample of N entangled states. We have observed that this quantity grows with the entanglement shared between Alice and Bob. This relation it is not one-to-one due to the fact that states with less entanglement but with more purity (Eve more disentangled) can be equally efficient. Nevertheless we have point out that, these states would be, inefficient in the distillation of the key. Finally, we would like to remark that our study is not restricted to quantum key distribution protocols, but can be extended to any other protocol that uses as a resource entangled continuous variables to establish a set of classically correlated bits between distant parties [15].

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